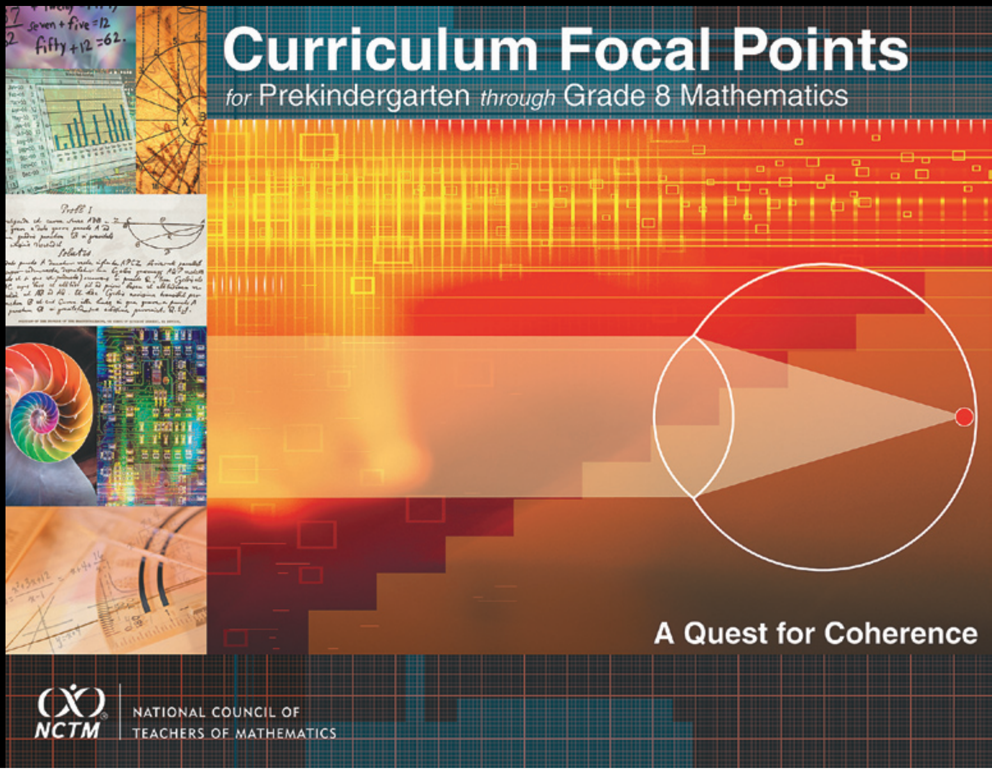
FOCAL POINTS— Grades 5 and 6



This is the fifth and final in a series of articles exploring the use of the 2006 National Council of Teachers of Mathematics' (NCTM's) publication Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence. *The series introduction by NCTM President Skip Fennell, explaining what the Curriculum Focal Points are and why NCTM developed them, appeared in the December 2007/January 2008 issue of* Teaching Children Mathematics *(page 315). In subsequent* TCM *articles, the authors of the various grade bands discuss the Focal Points for one or two grade levels. Because one principle of* Curriculum Focal Points *is that of cohesive curriculum, in which ideas develop across the grades, we encourage teachers of all grade levels to read the full series.*

Grade 5 and 6 students continue to explore concepts introduced in grades 3 and 4 but at a deeper level. For example, teachers introduce multidigit division, and students learn to perform basic operations with decimals and fractions, using like and unlike denominators. Number and operations concepts receive less emphasis, and geometric and algebraic concepts receive more emphasis.

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Fifth-Grade Mathematics

Division

Division is the subject of the first Focal Point in fifth grade. In third and fourth grades, students study division in connection with multiplication. They learn the meaning of division and how to represent it with groups and arrays. By studying patterns and relationships in basic multiplication facts and by relating multiplication and division, students can build a foundation for fluency with the basic multiplication and division facts.

In fifth grade, students develop an understanding of and fluency with multidigit division. In the standard algorithmic approach, the unknown factor (the quotient) is found place by place. "The standard algorithm" is the related set of ways to write this standard approach.

Students may benefit from initial work to build up the unknown factor more gradually. For example, to solve 434 ÷ 7, some students might "build up" multiples of 7 to 434, as shown in **figure 1a**, whereas other students might repeatedly subtract multiples of 7, as seen in **figure 1b.** To learn the standard algorithmic approach to division with understanding, children can learn to make simple area drawings and write the steps of the algorithm in the drawings or relate these steps to steps in the drawings. For example, to view the division problem 1721 ÷ 7 in terms of an area drawing, children will first need to understand that the solution to this division problem (the unknown factor) is the length of one side of a rectangle that has area 1721 square units and a side length of 7 units, as shown in **figure 2**. Using the standard algorithmic approach, students find the length of the unknown side in steps, starting with the highest place value and proceeding to places of lower value (Th = thousands; H = hundreds; T = tens).

Figure 3a indicates the steps in reasoning that students might use to find the length of the rectangle. Students can also record the place-value components of 1721 within the rectangle as 1000, 700, 20, and 1 instead of as 1 Th, 7 H, and 2 T, and 1, in which case their work will look like that in **figure 3b** or like Carla's method in **figure 1b**. Using this method, students may underguess or underestimate quotients, although it is important that students move toward using close factors and not continue to use the early methods shown in **figure 1a–b** (Agustin's and Carla's), where the build-up for each place is very slow.

Addition and subtraction of fractions and decimals

The second Focal Point in fifth grade concerns addition and subtraction of fractions and decimals. To understand addition and subtraction with fractions, students must understand that in a fraction,

Figure 1

Informal methods that students might use to solve 434 ÷ 7

(a) Agustin's method: building up multiples of 7 to 434

10 x 7 = 70
20 x 7 = 140
30 x 7 = 210
60 x 7 = 420
2 x 7 = 14

60 x 7	420
+2 x 7	+14
62	434

434 ÷ 7 = 62

(b) Carla's method: subtracting multiples of 7

434 ÷ 7 = 62

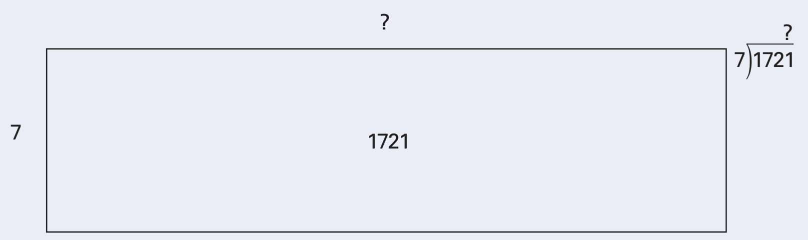
2		
10		
20		
20		
10		
7	434	
	–70	10 x 7
	364	
	–140	20 x 7
	224	
	–140	20 x 7
	84	
	–70	10 x 7
	14	
	–14	2 x 7
	0	

2		434 ÷ 7 = 62
10		
20	62	
20		
10		
7)434		
–70		10 x 7
364		
–140		20 x 7
224		
–140		20 x 7
84		
–70		10 x 7
14		
–14		2 x 7
0		

Figure 2

Viewing division as finding the length of a side of a rectangle

(a) The rectangle has an area of 1721 square units and a side length of 7 units:



(b) Break 1721 apart by place value to be able to work place by place:

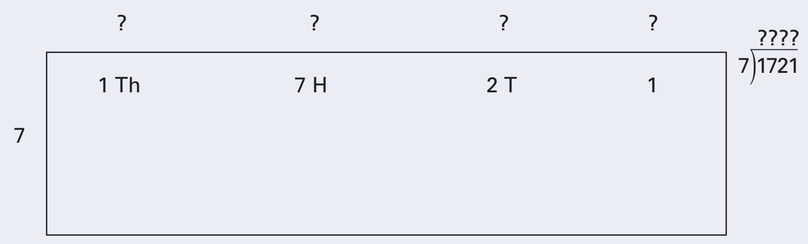
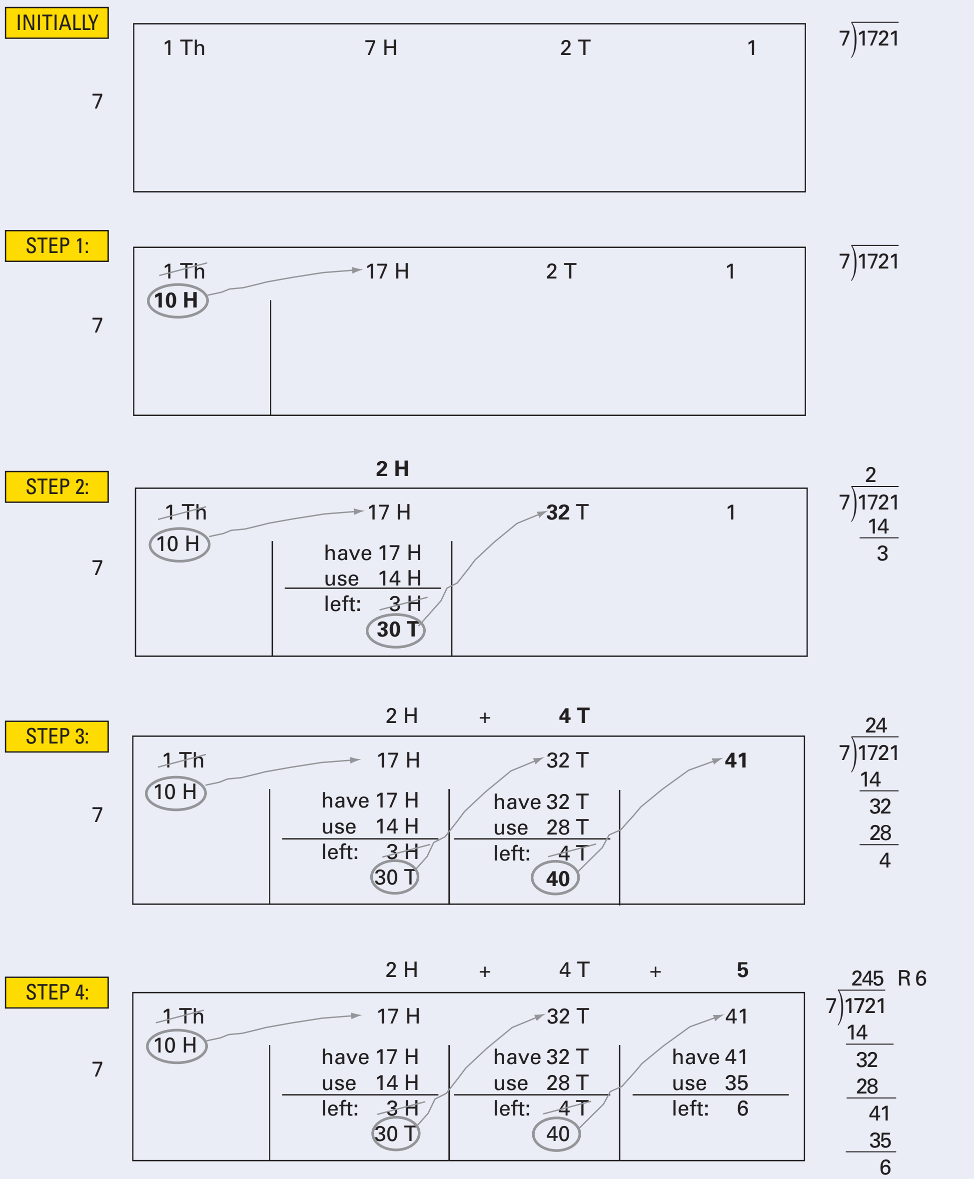
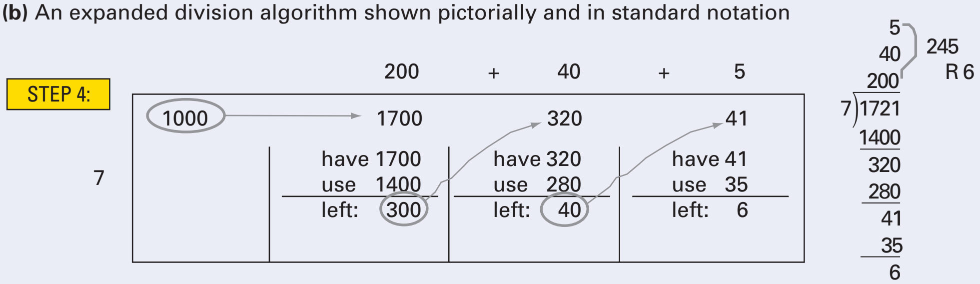


Figure 3

(a) The standard algorithmic approach recorded pictorially and numerically



(b) An expanded division algorithm shown pictorially and in standard notation



the numerator refers to the number of pieces and the denominator refers to the size of the pieces. For example, 2/9 stands for 2 pieces, each of which is 1/9 of a whole. Adding or subtracting fractions that have the same denominator is just a matter of adding or subtracting the number of pieces the fractions refer to, as indicated in **figure 4**.

As students learned in fourth grade, every fraction is equivalent to many other fractions. If two fractions do not have the same denominator, then students should look for equivalent fractions that do have the same denominator. Once students find equivalent fractions that have the same denominator, they can add or subtract the fractions as before.

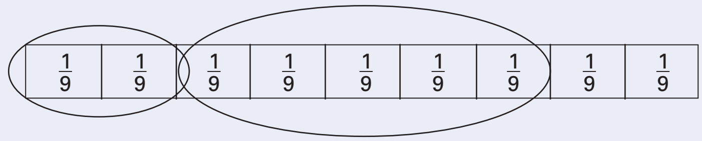
Drawing simple pictures can help students when they solve story problems that involve adding and subtracting fractions. For example, consider this problem:

> Carlton had $72. Then he spent 1/4 of his money on books and 3/8 of his money on sports equipment. How much money did Carlton have left?

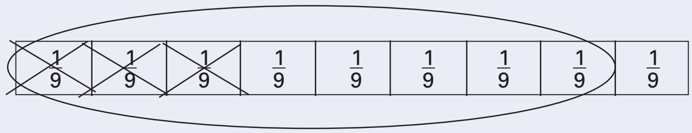
To help them solve this problem, students might draw a simple picture. **Figure 5** illustrates the steps that a student might take.

Figure 4

Adding and subtracting fractions with like denominators



$\frac{2}{9} + \frac{5}{9} = \frac{2+5}{9} = \frac{7}{9}$ 2 ninths + 5 ninths = 7 ninths



$\frac{8}{9} - \frac{3}{9} = \frac{8-3}{9} = \frac{5}{9}$ 8 ninths – 3 ninths = 5 ninths

Figure 5

Using simple pictures to aid in solving a fraction problem

Area and volume

The third Focal Point at fifth grade concerns area as well as three-dimensional shapes and their volume and surface area. This work extends students' fourth-grade work on area.

In fourth grade, students study areas of rectangles. Students decompose rectangles to help them understand the multiplication algorithm. In fifth grade, students decompose and rearrange other shapes to determine their areas. An especially useful technique for determining areas and developing area formulas is to relate the area of a shape to the area of a rectangle. For example, as indicated in **figure 6**, students could cut a parallelogram apart and rearrange the pieces into a rectangle that has the same base and height. By applying the method of decomposing and rearranging to various parallelograms, students can explain why the area of a parallelogram is the base times the height.

Similarly, the area of a triangle can also be related to the area of a rectangle by decomposing and combining, as shown for a right triangle in **figure 7a**. By working with various triangles, students can see why the area of a triangle is base × height ÷ 2. The method of decomposing shapes is useful not only for deriving area formulas but also for solving other area problems. **Figure 7b** illustrates several challenging area problems that students can solve by using a combination of area formulas and decomposing.

Fifth graders also compose and decompose three-dimensional shapes. They consider how to make these shapes by making nets (patterns) and consider their component parts, such as faces, edges, and vertices. Students understand that the

Figure 6

Finding the area of a parallelogram by decomposing and rearranging it as a rectangle

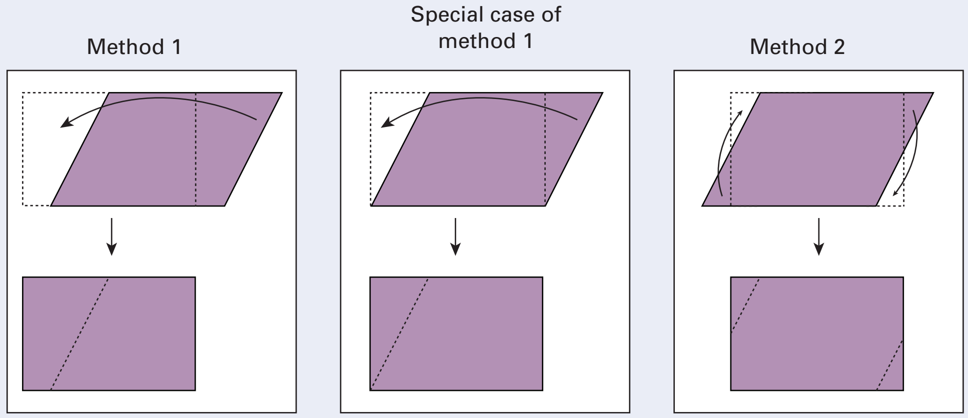
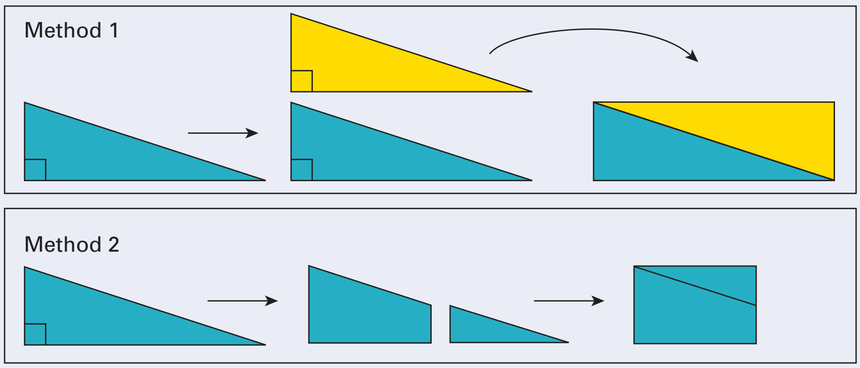
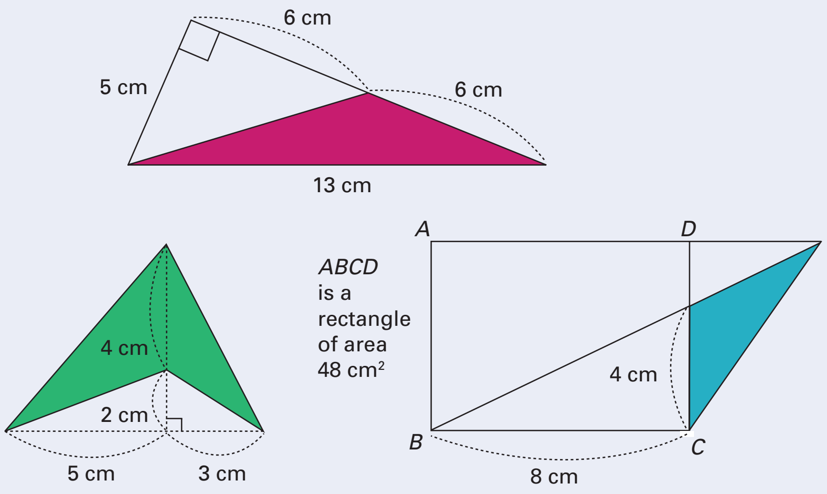


Figure 7

(a) Methods for finding the area of a right triangle



(b) Students can be asked to determine the areas of the shaded shapes.



volume of an object in cubic units is the number of 1-unit-by-1-unit-by-1-unit cubes it takes to make (or fill) the object—with the understanding that cubes may need to be cut apart and possibly rearranged. By viewing rectangular prisms made of layers (as in **fig. 8**), students determine that the volume of a rectangular prism is the area of the base (which is the number of cubes in a layer) times the height.

Sixth-Grade Mathematics

Multiplication and division of fractions and decimals

In the first Focal Point on number and operations, students extend their work on multiplication and division of whole numbers in previous grades to multiplication and division of fractions and decimals.

Students can use a combination of story problems and drawings to help them understand why it makes sense to multiply the numerators and the denominators when multiplying fractions. For example, if it takes 2 cups of flour to make a recipe, then how much flour will you need to make 3 batches of, or triple, the recipe? You will need 3 × 2 = 6 cups of flour. Likewise, if it takes 4/5 of a cup of flour to make a recipe, then how much flour will you need to make 2/3 of the recipe? You will need 2/3 × 4/5 cups of flour. But what amount of flour is this? Students can use pictures like the one in **figure 9** to explain why the amount of flour needed is (2 × 4)/(3 × 5) = 8/15 of a cup of flour.

Students can also use a combination of story problems, pictures, and double number lines to help them understand and reason about fraction division. For example, if 6 pounds of nails fill 3 identical containers, then how many pounds of nails fill one container? To solve this problem, students calculate 6 ÷ 3 = 2. Likewise, if 3/4 pound of nails fills a container 2/3 full, then how many pounds of nails will fill the container? This is the same problem as the previous one except that the numbers have changed, so students should see that this problem can also be solved by division, namely, by calculating 3/4 ÷ 2/3 (or they may view the problem as ([2/3])x × ? = 3/4). To find the solution to this division problem, students can use pictures, a double number line, or a combination of both as shown in **figure 10a**. To understand why the solution to the division problem can be obtained by multiplying the dividend by the reciprocal of the divisor, namely by calculating 3/4 × 3/2, students could use the reasoning indicated in **figure 10b**.

Figure 8

Viewing a rectangular prism in layers to find the volume

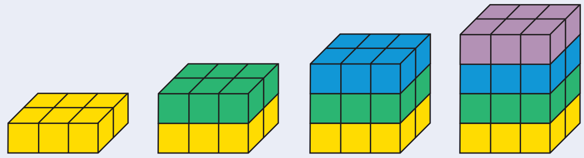
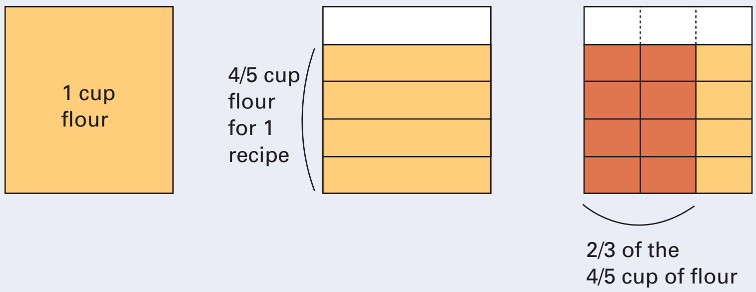


Figure 9

Explaining fraction multiplication with aid of a picture



Decimal multiplication and division relies on whole-number multiplication and division, together with correct placement of the decimal point in the answer. Students should understand why we place the decimal point where we do. **Figure 11** indicates a line of reasoning for the placement of the decimal point in the solution to 3.84 × 1.2. Students should also be able to use estimation to determine the placement of the decimal point in this problem. Namely, since 3.84 is between 3 and 4 and 1.2 is between 1 and 2, the solution to 3.84 × 1.2 must be between 3 and 8, so the only reasonable location for the decimal point in 4608 is after the 4, namely, 4.608.

Students can relate both decimal and fraction division to whole-number division by changing the unit so that the divisor and dividend become whole numbers. For example, students could compare the following problems:

Problem 1. There is 6/8 of a pizza left. How many people can eat pizza if each person will eat 2/8 of a pizza?

Problem 2. Maya has $0.60. Each sticker costs $0.20. How many stickers can Maya buy?

Problem 3. Benton has $6. Each book costs $2. How many books can Benton buy?

Figure 10

(a) A double number line and picture for fraction division

(b) Using a double number line to reason about fraction division

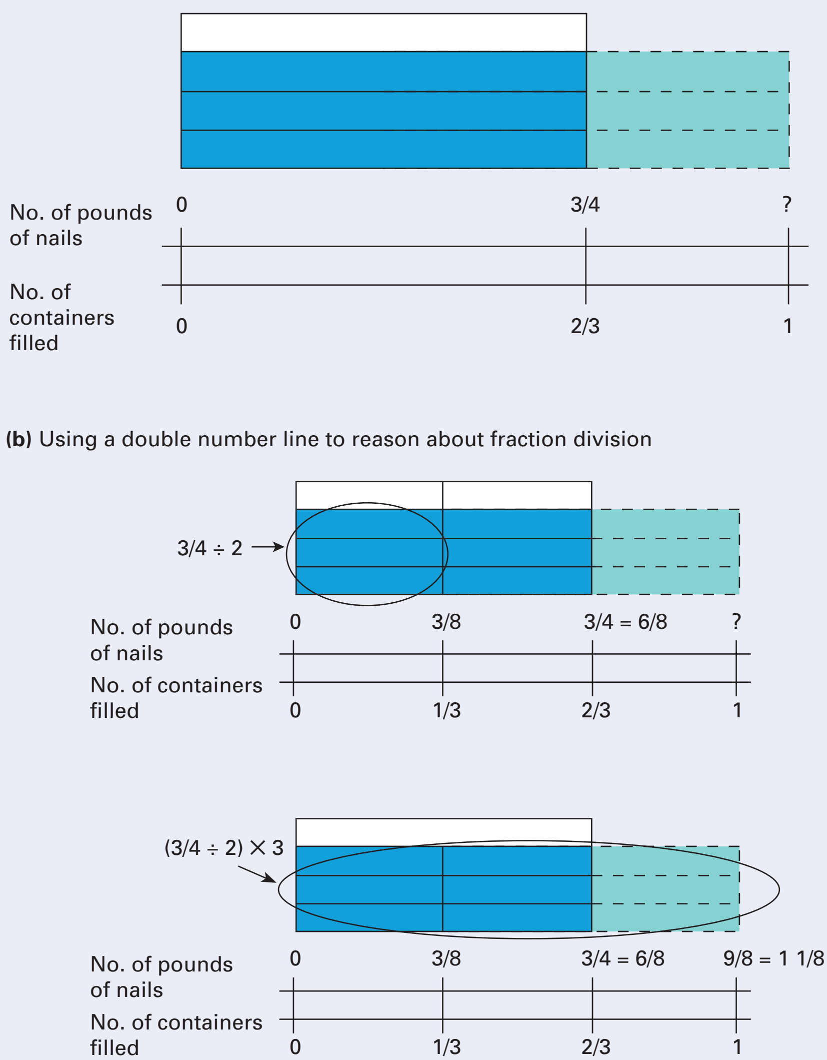


Figure 11

Reasoning about the placement of the decimal point in decimal multiplication

3.84	× 100 →	384
× 1.2	× 10 →	× 12
768		768
3840	← ÷ 100 ÷ 10	3840
4.608		4608

All three problems can be solved by calculating 6 ÷ 2, because the first problem can be rephrased in terms of 6 eighths and 2 eighths of a pizza and the second problem can be rephrased in terms of 6 tenths and 2 tenths of a dollar. In general, a decimal division problem, such as 5.6 ÷ 0.07, can be viewed as asking, “How many 0.07s are in 5.6?” or “How many 7 hundredths are in 560 hundredths?” which is equivalent to “How many 7s are in 560?” or 560 ÷ 7.

Ratio and rate

The second Focal Point at sixth grade connects ratio and rate to multiplication and division. For example, if students are considering mixing blue and yellow paint in the ratio of 2 to 3 to make a certain shade of green, they could create a table to show several different amounts of paint to mix in that ratio (see **fig. 12a**).

From the table, students can see that to make 35 pails of the green paint, they will mix 14 pails of blue and 21 pails of yellow. The table can connect to multiplication because the rows are produced by multiplication. Several other interesting connections can also be made. The table can also connect to equivalent fractions. For example, in the second and third rows, we see the equivalent fractions

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{14}{21},$$

which occur because every entry in the second row is 2/3 of the corresponding entry in the third row. If students graph the relationship between the blue paint and the yellow paint, they may be surprised and delighted to notice that it is a straight line (see **fig. 12b**).

By reasoning about multiplication and division in a ratio table, students can solve problems about ratios. For example, consider again the green paint that is made by mixing blue and yellow paint in a ratio of 2 to 3. How many pails of blue paint (and how many pails of yellow paint) will be needed to make 100 pails of green paint? (See **fig. 13a**.) Recognizing that the entries in each column of the ratio table are a multiple of the entries in the first column will allow students to use multiplication and division to solve the proportion, as indicated in **figure 13b**.

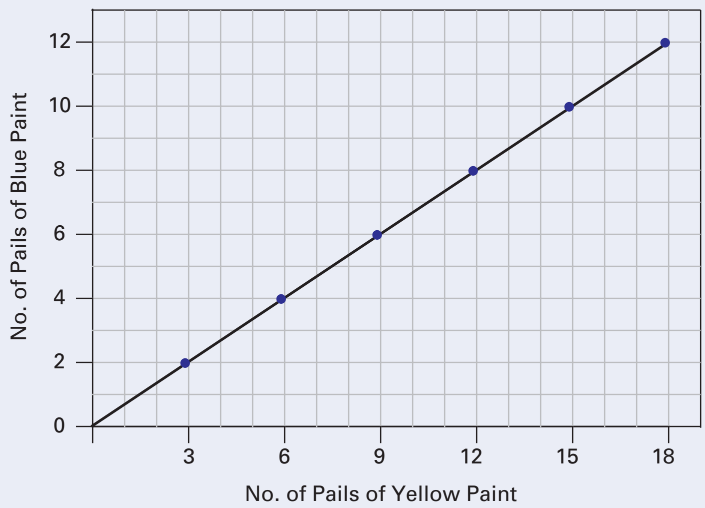
Students can also solve problems about ratios by analyzing simple drawings that indicate the relative sizes of quantities. Consider again the case of green paint that is made by mixing blue and yellow paint in the ratio of 2 to 3. Students can use a simple drawing, such as the one in **figure 14a**, to represent this ratio. Students can use these simple quantity drawings to solve problems. For example, **figure**

Figure 12

(a) A table of equivalent ratios

No. of batches	1	2	3	4	5	6	7
No. of pails of blue paint	2	4	6	8	10	12	14
No. of pails of yellow paint	3	6	9	12	15	18	21
No. of pails of green paint produced	5	10	15	20	25	30	35

(b) A graph of equivalent ratios



14b shows how students might use the drawing to help determine how much blue and yellow paint will be needed to make 30 pails of the green paint.

By solving ratio problems with the aid of ratio tables and simple drawings, students have the opportunity to reason about multiplication and division and develop their sense of ratio and proportion before they learn to use more abstract methods.

Expressions and equations

The Algebra focus in sixth grade is on writing, interpreting, and using mathematical expressions and equations. For example, when blue and yellow paint is mixed in the ratio of 2 to 3 to create a certain shade of green paint, students can let b stand for a number of pails of blue paint and y stand for a number of pails of yellow paint. Then students can use a ratio table or a simple quantity drawing to determine an equation that relates b and y. For example, students might notice that $b = 2/3 \times y$ or that $y = 3/2 \times b$, or they might notice that $3b = 2y$ because in the quantity drawing, 3 multiples of blue are as long as 2 multiples of yellow (see **fig. 15**). It is important for students to understand that b and y stand for *numbers* of pails of paint and attend to the relationship beween the two numbers. Imprecise wording, such as "2 blues equal 3 yellows," can lead to the *incorrect* equation $2b = 3y$.

Students can also relate simple quantity drawings to the standard algebraic methods for solving problems. For example, consider this basketball problem:

> When they were playing basketball, Shauntay got 3 times as many baskets as Jessica. Together, Shauntay and Jessica got 36 baskets. How many baskets did Jessica get?

To solve this problem using standard algebraic methods, students can let n be the number of baskets that Jessica got. Then Shauntay got $3n$ baskets, and together both girls got $n + 3n = 4n$ baskets. Since $4n = 36$, $n = 36 \div 4 = 9$; so, Jessica got 9 baskets (and Shauntay got 27). Students can use a simple quantity drawing, such as that in **figure 16**, to help them with this line of reasoning. If students have used quantity drawings for single-step problems in earlier grades, such multiple-step problems will be more accessible. It is particularly important

Figure 13

(a) Can students use a ratio table to solve a proportion?

No. of batches	1	2	3	4	5	
No. of pails of blue paint	2	4	6	8	10	?
No. of pails of yellow paint	3	6	9	12	15	?
No. of pails of green paint produced	5	10	15	20	25	100

$$\frac{2}{5} = \frac{?}{100}$$

(b) Reasoning about a ratio table to solve a proportion

Step 1

No. of batches	1	2	3	4	5	
No. of pails of blue paint	2	4	6	8	10	?
No. of pails of yellow paint	3	6	9	12	15	?
No. of pails of green paint produced	5	10	15	20	25	100

× 20

$$\frac{2}{5} = \frac{?}{100}$$

× 20

× 20

Step 2

No. of batches	1	2	3	4	5	20
No. of pails of blue paint	2	4	6	8	10	40
No. of pails of yellow paint	3	6	9	12	15	60
No. of pails of green paint produced	5	10	15	20	25	100

× 20

× 20

$$\frac{2}{5} = \frac{40}{100}$$

× 20

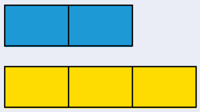
for students to have experienced such problem situations as this one: "Erica had some money in her bank. Her uncle gave her $27. Now she has $82. How much did she have to start?" For these problems, students need to represent the situation (e.g., $s + 27 = 82$) and then reflect on it to find the solution (by adding on tens and ones from 27 to make 82 or by subtracting 27 from 82).

Focal Points: A Foundation

Curriculum Focal Points describes the key mathematical ideas and skills that are essential for further study in mathematics. The algebra, geometry, measurement, and data analysis and probability that students study in seventh and eighth grades and in high school rely on the foundational concepts and skills of the grades pre-K–6 Focal Points. In particular, without a good understanding of the kinds of problems that addition, subtraction, multiplication, and division solve and without fluency with these operations, students' progress will be limited. It is therefore vital to focus on and give sufficient time at each grade level for the Focal Points at that grade. Doing so will allow students to understand and become fluent in topics that are at the core of mathematics and that enable further progress in mathematics.

Figure 14

(a) Representing the ratio 2 to 3 of blue paint to yellow paint



(b) Reasoning about a quantity diagram to solve a ratio problem

5 equal parts make 30 pails
$30 \div 5 = 6$ pails in each part

$2 \times 6 = 12$ pails blue paint
$3 \times 6 = 18$ pails yellow paint
30 pails green paint

Figure 15

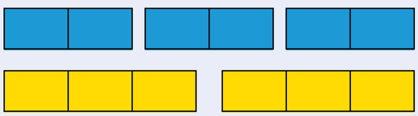
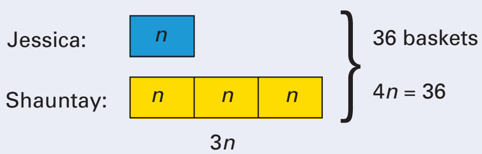


Figure 16

Relating a simple quantity drawing to standard algebraic methods



NCTM's Focal Points help us reorganize what often have been long, unorganized lists of specific skills in various standards. The Focal Points help us see important mathematical ideas as interconnected clusters of *related* ideas and skills. For example, by asking students to think about ratio and rate in terms of multiplication and division, teachers can bring out connections to the multiplication table, equivalent fractions, and graphs of lines. The skill that students develop when they explain why the area formula for triangles is valid by cutting apart and rearranging pieces can also be used to solve other problems about area and volume.

Bibliography

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Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence is available without charge for PDF download as a full document or by section at www.nctm.org/standards/focalpoints.aspx?id=282. The published document is available for sale through the NCTM catalog. Readers may also be interested in a similar series that began in the August 2007 issue of *Mathematics Teaching in the Middle School* and ran through the December 2007/January 2008 issue. ▲